Reaching Zero Rapidly

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Abstract

The controlled process $X(t)$ with values in $(-\infty,0]$ is given by a stochastic differential equation

$$dX(t) = \mu(t)dt + \sigma(t)dW_t, \qquad X_1(0)=x$$

where the non-anticipative controls μ and σ are to be chosen so that $(\mu(t),\sigma(t))$ remains in a given set $\mathcal{S}$. The object is to maximize (minimize) the expectation of β^T where $0<\beta<1$ ($\beta>1$) and T is the hitting time of zero. A complete solution is given for any $\mathcal{S}$, and an application is made to continuous-time red-and-black.

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0. Introduction

Consider a process $\{X_1(t)\}$ on $(-\infty,0]$ given by a stochastic differential equation

$$dX_1(t) = \mu(t)dt + \sigma(t)dW_t, \quad X_1(0)=x_1$$

where $\{W_t\}$ is standard Brownian motion and $\{\mu(t)\}$ and $\{\sigma(t)\}$ are non-anticipative controls to be chosen so that $(\mu(t),\sigma(t))$ remains in a specified set $\mathcal{S}$. Let T be the first time X_1 reaches the origin. In [2] the problems of minimizing or maximizing the expected value of T were solved for any $\mathcal{S}$; the value function and optimal strategies were given explicitly in terms of $\mathcal{S}$. Minimizing the expected value of T is a natural criterion for getting to the origin rapidly. However, there are other criteria, two of which are considered here.

Our first problem is to maximize the expected value of β^T, where β is a positive constant less than one. The second problem is to minimize the expected value of β^T where now the constant β is chosen greater than one. In both cases we obtain a complete solution, for arbitrary $\mathcal{S}$.

The problems are of unequal difficulty. The first problem is actually quite easy. The reason the second problem is harder is that in the solution one must distinguish between the two cases $I<\infty$ and $I=\infty$, where the quantity I is defined by

$$I = \inf_{\varepsilon>0} \sup \{\mu+\varepsilon\sigma^2 : (\mu,\sigma) \in \mathcal{S}\} .$$

As is common in control theory problems, we obtain a natural candidate $Q(x_1)$ for a value function and try to prove it correct by applying an appropriate verification theorem. As it turns out, Q is indeed the correct value function if and only if $I<\infty$. Hence there must be something in the application of the verification theorem distinguishing between $I<\infty$ and $I=\infty$, and this indicates we should expect technical difficulties. The $I<\infty$, $I=\infty$ dichotomy also appeared in [2], but in the present work more delicate constructions for overcoming the obstacles are required.

The solutions of the two general problems make it possible for us to solve the particular problems of discounted ($0<\beta<1$) and inflated ($\beta>1$) red-and-black. It is shown here for the continuous-time problem, as it was by Klugman [3] in discrete-time, that bold play is optimal for subfair, discounted red-and-black. The superfair case is also explicitly solved here for the continuous-time problem although it remains open in discrete-time.

1. Preliminaries.

We begin by explaining the continuous time gambling set-up of [4] and [2].

A continuous-time gambling problem is a triple (F,Σ,u) where

(1.1) the state space F is a Borel subset of the Euclidean space $\mathbb{R}^d$ having non-empty interior,

(1.2) the gambling house Σ is a mapping which assigns to each $x \in F$ a non-empty collection $\Sigma(x)$ of processes $X =$

$\{X_t, t\geq 0\}$ with state space F such that $X_0 = x$ and X has right-continuous paths with left-limits,

(1.3) the utility function u is a Borel function from F to the real line.

A process $X \in \Sigma(x)$ is said to be available at x. Each available X is defined on some probability space $(\Omega,\mathcal{F},P)$ and is adapted to an increasing filtration $(\mathcal{F}_t, t\geq 0)$ of complete sub-sigma fields of $\mathcal{F}$. The probability space and filtration may depend on X.

A player, starting at position $x \in F$, selects a process $X \in \Sigma(x)$ and receives payoff $u(X)$ defined by

$$u(X) = E[\limsup_{t\to\infty} u(X_t)]. \tag{1.4}$$

The expectation occurring on the right is assumed to be well-defined for every available process X.

The value function V is defined by

$$V(x) = \sup\{u(X): X \in \Sigma(x)\}$$

for every $x \in F$. A process $X \in \Sigma(x)$ is optimal at x if

$$u(X) = V(x).$$

From now on, each process $X = \{X_t\}$ under consideration will be an Ito process of the form

$$X_t = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW_s, \tag{1.5}$$

where $W = \{W_t\}$ is a standard m-dimensional Brownian motion process on $(\Omega,\mathcal{F},P)$ adapted to increasing, right-continuous σ-fields $\{\mathcal{F}_t\}$ and $\mathcal{F}_t$ is independent of $\{W_{t+s}-W_t,\ s\geq 0\}$. The function $\alpha = \alpha(t,\omega)$ is to be $\mathbb{R}^d$-valued, progressively measurable, and such that

$$(1.6) \qquad \int_0^t |\alpha(s)|ds < \infty \quad \text{a.s.} \quad \text{for all } t.$$

The function $\beta = \beta(t,\omega)$ has as values real $d\times m$ matrices, is progressively measurable, and satisfies

$$(1.7) \qquad \int_0^t |\beta(s)|^2 ds < \infty \quad \text{a.s.} \quad \text{for all } t.$$

For each pair (a,b), where $a \in \mathbb{R}^d$ is a $d\times 1$ vector and b is a $d\times m$ real-valued matrix, define the differential operator $D(a,b)$ for sufficiently smooth functions $Q:\mathbb{R}^d \longrightarrow \mathbb{R}$ by

$$D(a,b)Q(y) = Q_x(y)a + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} Q_{x_i x_j}(y)(bb')_{ij}$$

where

$$Q_x(y) = \left(\frac{\partial Q}{\partial x_1},\ldots,\frac{\partial Q}{\partial x_d}\right),$$

$$Q_{x_i x_j} = \frac{\partial^2 Q}{\partial x_i \partial x_j},$$

and b' is the transpose of b.

We now specify $\Sigma(x)$ by specifying the possible values of α and β. To this end, let $C(x)$ be, for each $x \in F$, a non-empty set of pairs (a,b), where $a \in \mathbb{R}^d$ and b is a real $d\times m$ matrix. (The idea is that $C(x)$ is the set from which a player at state x may choose the value of (α,β).) Assume also that every available process X is absorbed at the time T_X of its first exit from F°, the interior of F. These conditions define a function Σ_C on F where $\Sigma_C(x)$ is the collection of all processes X having paths in F

and satisfying (1.5), (1.6), and (1.7) together with

(1.8) $\quad (\alpha(t,\omega),\beta(t,\omega)) \in C(X_t(\omega))$ for all (t,ω),

(1.9) $\quad (\alpha(t,\omega),\beta(t,\omega)) = (0,0)$ for $t \geq T_X(\omega)$,

(1.10) $\quad C(x) = \{(0,0)\}$ for $x \in F-F^\circ$.

Let Σ be a gambling house such that $\Sigma(x) \subset \Sigma_C(x)$ for every $x \in F$. The following proposition is related to other verification lemmas in [2] and [4]. Since the hypotheses here differ slightly from those in [2] and [4], we provide a proof.

<u>Proposition 1.1</u> Let G be an open subset of $\mathbb{R}^d$ which contains F. Suppose $Q:G \to \mathbb{R}$ has continuous second order derivatives on G and that for every $x \in F^\circ$ and every $X \in \Sigma(x)$,

(i) $E[\limsup_{t\to\infty} Q(X_t)] \geq E[\limsup_{t\to\infty} u(X_t)]$,

(ii) $P[D(\alpha(t),\beta(t))Q(X_t) \leq 0$ for all $t \geq 0] = 1$, where α and β are related to X as in (1.5),

(iii) there exists an integrable random variable Y such that for all $t\geq 0$, $Q(X_t) \geq Y$.

Then $Q \geq V$.

<u>Proof</u>. Let $x_0 \in F$ and $X \in \Sigma(x_0)$. For each $t\geq 0$, Ito's Lemma gives

(1.11) $\quad Q(X_t) = Q(x_0) - A_t + M_t,$

where

$$A_t = -\int_0^t D(\alpha(s),\beta(s))Q(X_s)ds$$

and

$$M_t = \int_0^t Q_x(X_s)\beta(s)dW_s .$$

Notice that by conditions (ii) and (iii),

$$M_t = Q(X_t) - Q(x_0) + A_t \geq Y - Q(x_0)$$

holds for all $t \geq 0$ with probability one. Therefore the local martingale M_t is a supermartingale (see [1], VI.29), and if τ is any almost-surely finite stopping time,

$$(1.12) \qquad EM_\tau \leq EM_0 = 0.$$

From (1.11), (1.12) and condition (ii),

$$EQ(X_\tau) \leq Q(x_0) .$$

By condition (i) and by Lemma 1 of [4], it follows that $Q \geq V$. ∎

2. Maximizing $E[\beta^T]$, $0<\beta<1$.

We formalize the first problem from the introduction as a continuous-time gambling problem in $\mathbb{R}^2$. The first coordinate x_1 will be constrained to $(-\infty,0]$ and indicates the player's position, while the second coordinate x_2 in $(-\infty,\infty)$ increases at a constant rate one and merely keeps track of the time. Define $F = \{x \in \mathbb{R}^2: x=(x_1,x_2), -\infty<x_1\leq 0\}$. By our conventions each process X available at x will be absorbed at $T = T_X = \inf\{t: X_1(t)=0\}$.

Let $\mathcal{S} \subseteq \mathbb{R}\times[0,\infty)$ and

$$C_0 = \left\{\begin{bmatrix}\mu\\1\end{bmatrix}, \begin{bmatrix}\sigma\\0\end{bmatrix} : (\mu,\sigma) \in \mathcal{S}\right\} \tag{2.1}$$

and for every x in the interior of F, $C(x) = C_0$. Every $X\in\Sigma_C(x)$ can be specified by stochastic differential equations

$$\begin{aligned} dX_1(t) &= \mu(t)dt + \sigma(t)dW_t \\ dX_2(t) &= dt \\ X_1(0) &= x_1,\ X_2(0) = x_2, \end{aligned} \tag{2.2}$$

where μ and σ are progressively measurable and $(\mu(t),\sigma(t)) \in \mathcal{S}$, $t<T$, $X(t) = X(T)$ for $t\geq T$.

Our utility function will be

$$u(x) = \beta^{x_2}. \tag{2.3}$$

Note that for $T<\infty$,

$$X_1(T) = 0, \quad X_2(T) = T + X_2(0)$$

and so

$$u(X) = E\left[\beta^{T+x_2}\right]. \tag{2.4}$$

Here we interpret β^{T+x_2} as 0 if $T = \infty$. Now let $\Sigma(x) = \Sigma_C(x)$. The value function $V(x_1,x_2)$ must clearly have the form

$$V(x_1,x_2) = \beta^{x_2}V(x_1). \tag{2.5}$$

The form of $V(x_1)$ is also easily obtained. Given $X \in \Sigma(x)$ and $y \in [x,0]$, let T_{xy} be the first time X attains y. Consider the problem of maximizing $E[\beta^{T_{xy}}]$ and denote the corresponding value function by V_{xy}. One easily sees $V_{xy} = V(x-y)$. It now appears that $T_{x0} = T_{xy} + T_{y0}$ and hence a stopping time argument gives $V(x) = V(x-y) \cdot V(y)$, leading to

$$V(x_1) = e^{\lambda x_1}, \quad \lambda \geq 0. \tag{2.6}$$

Also, since the problem of minimizing $E[\beta^{T_{xy}}]$ looks the same as that of minimizing $E[\beta^{T_{x-y}}]$, one expects to obtain optimal strategies of the form $\mu(x) \equiv \mu$ and $a(x) \equiv a$, where we write $a(x) = \sigma^2(x)$ and μ and a will be constants depending on $\mathcal{S}$. If $a=0$ and $\mu \leq 0$, then the process reaches zero with probability zero. So assume the maximum of a and μ is positive. For such a constant strategy the expected payoff is $W(x_1,x_2) = \beta^{x_2} W(x_1)$, with $W(x_1)$ satisfying

$$\tfrac{1}{2}aW'' + \mu W' + (\log\beta)W = 0, \quad W(0) = 1.$$

But again W should be an exponential $e^{\lambda x_1}$ with $\lambda \geq 0$, and this implies

$$W(x_1) = e^{\lambda x_1}, \tag{2.7}$$

where

$$(2.8) \qquad \lambda = \lambda(\mu,a) = \begin{cases} \dfrac{-\mu + \sqrt{\mu^2 - 2a\log\beta}}{a} & a>0 \\[2ex] \dfrac{-(\log\beta)}{\mu} & a=0,\ \mu>0. \end{cases}$$

Notice that in the case $a>0$, λ is the positive root of

$$(2.9) \qquad \tfrac{1}{2}a\lambda^2 + \mu\lambda + \log\beta = 0.$$

For fixed λ, the relation between λ, μ, and a in (2.8) defines a line in the (μ,a)-plane, namely

$$(2.10) \qquad \ell_\lambda : a = \frac{-2}{\lambda}\mu - \frac{2\log\beta}{\lambda^2}.$$

For our purposes, only

$$(2.11) \qquad \bar{\ell}_\lambda = \ell_\lambda \cap \{(\mu,a) : a \geq 0\}$$

is relevant. It now appears that to optimize the expected payoff in (2.7) we want to use strategies whose (μ,a)-pairs lie on $\bar{\ell}_{\lambda^*}$, where

$$(2.12) \qquad \lambda^* = \inf\{\lambda(\mu,\sigma^2) : (\mu,\sigma)\in\mathcal{S},\ \max(\sigma,\mu)>0\}.$$

Theorem 2.1: If $\mathcal{S} \subseteq (-\infty,0]\times\{0\}$, then $V(x_1,x_2) = 0$. Otherwise,

$$(2.13) \qquad V(x_1,x_2) = \beta^{x_2} e^{\lambda^* x_1}$$

with λ^* as defined in (2.12).

Proof. The first assertion of the theorem is obvious. Assume then that $\mathcal{S}$ contains a point (μ,σ) with $\max(\mu,\sigma^2)>0$. Let $Q(x_1,x_2)$

$= \beta^{x_2} e^{\lambda^* x_1}$. By (2.5) and (2.7) one can realize expected payoffs arbitrarily close to $Q(x_1,x_2)$, so that $Q(x_1,x_2) \le V(x_1,x_2)$. For the opposite inequality use Proposition 1.1. It must be checked that Q satisfies the conditions (i)-(iii). Condition (iii) is immediate. Condition (i) follows from the fact that for every available process $X = \{X_t\}$,

$$\limsup_{t\to\infty} Q(X_t) = \limsup_{t\to\infty} u(X_t) = \begin{cases} \beta^{T+x_2} & \text{if } T<\infty \\ 0 & \text{if } T=\infty. \end{cases}$$

Condition (ii) follows once we have checked

$$\tag{2.14} \tfrac{1}{2}\sigma^2\left[\lambda^*\right]^2 + \mu\lambda^* + \log\beta \le 0, \qquad (\mu,\sigma)\in\mathcal{S}.$$

In the case $\sigma^2=0$, $\mu=0$, this is immediate from the definitions of $\lambda(\mu,\sigma^2)$ and λ^*. In the case $\sigma^2>0$, (2.14) follows again from these definitions together with the fact that λ^* lies between 0 and $\lambda = \lambda(\mu,\sigma^2)$, which is the positive root of the equation (2.9). ■

<u>Example</u>. Discounted, continuous-time red-and-black. Suppose an investor has initial fortune y, $0 < y < 1$, and seeks to attain a fortune of 1. If $Y(t)$ is the player's fortune at time $t \ge 0$, he can invest $s(t)Y(t)$, $0 \le s(t) \le 1$, in a venture with rate of return μ_0 and standard deviation $\sigma_0>0$. More formally, the process $Y(t)$ is given by a stochastic differential equation

$$dY(t) = s(t)Y(t)[\mu_0 dt + \sigma_0 dW_t], \qquad Y(0) = y$$

where $s(t)$ is a non-anticipative function such that $0 \leq s(t) \leq 1$. If the object is to maximize the probability of reaching 1, then (cf. [4],[5]) <u>bold play</u>, for which $s(t) \equiv 1$, is optimal in the <u>subfair case</u> ($\mu_0 \leq 0$), and <u>proportional play</u>, for which $s(t) \equiv c$ $(0 < c < 2\mu_0\sigma_0^{-2})$, is optimal in the <u>superfair case</u> ($\mu_0 > 0$). Here the value of the goal is discounted at rate β and the object is to maximize $E\beta^T$, where $T = \inf\{t \geq 0: Y(t) = 1\}$. The problem can be reduced to a special case of the problem of this section by the change of coordinates

(2.15) $$X_1(t) = \log Y(t).$$

By Ito's formula,

$$dX_1(t) = (s(t)\mu_0 - \tfrac{1}{2}s(t)^2\sigma_0^2)dt + s(t)\sigma_0 dW_t.$$

The control set $\mathcal{S}$ is now a one-parameter family

(2.16) $$\mathcal{S} = \{(s\mu_0 - \tfrac{1}{2}s^2\sigma_0^2,\ s\sigma_0) : 0 \leq s \leq 1\},$$

and formula (2.8) can be written in the form

(2.17) $$\lambda(\mu,\sigma^2) = \lambda(s)$$

$$= \frac{-(\mu_0 - \frac{1}{2}s\sigma_0^2) + \sqrt{q(s)}}{s\sigma_0^2} \quad \text{if } s > 0$$

where $q(s) = (\mu_0 - \frac{1}{2}s\sigma_0^2)^2 - 2\sigma_0^2 \log \beta$. After some algebra it follows from (2.17) that, for $\mu_0 \leq 0$, $\lambda'(s) \leq 0$ on $(0,1]$ and, hence, the infimum $\lambda^* = \lambda(1)$. So bold play is again optimal in the subfair case. Consider next the superfair case $\mu_0 > 0$.

Define the number

$$c = c(\mu_0, \sigma_0, \beta) = \mu_0/\sigma_0^2 - 2(\log \beta)/\mu_0.$$

More algebra shows that $\lambda'(s) > 0$ if and only if $s >$ maximum$(c,0)$. Thus if $c \geq 1$, λ is decreasing on $(0,1]$ and bold play is optimal yet again. If $0 < c < 1$, an optimal strategy is given by $s(t) \equiv c$. If $c \leq 0$, there is no optimal strategy, but $s(t) \equiv \varepsilon$ for small positive ε will be almost optimal. An analogous problem in discrete-time was solved by Klugman [3] in the subfair case, but the discrete-time superfair problem remains open.

3. Minimizing $E[\beta^T]$, $1<\beta$.

Again we fit the problem into the set-up of Section 1. Since we discussed maximization problems there, we will seek to maximize $-\beta^T$. That is, we proceed exactly as in Section 2, assuming (2.1), (2.2) but now defining

$$u(x) = -\beta^{x_2}, \tag{3.1}$$

so that $u(X) = -E\left[\beta^{T+x_2}\right]$. Let

$$\Sigma(x) = \{X \in \Sigma_C(x) \colon u(X) > -\infty\} = \{X \in \Sigma_C(x) \colon E\beta^T < \infty\},$$

and we have

$$V(x_1, x_2) = \beta^{x_2} V(x_1). \tag{3.2}$$

The arguments that lead to (2.6) now give

$$V(x_1) = -e^{\lambda x_1}, \ \lambda \le 0. \tag{3.3}$$

Again let us consider the expected payoff $W(x_1,x_2)$ for a constant strategy $\mu(t)\equiv\mu$ and $\sigma(t)\equiv\sigma$, $\sigma^2=a$. If $a=0$ and $\mu>0$, obviously

$$W(x_1,x_2) = -\beta^{x_2} e^{\lambda x_1}, \tag{3.4}$$

where $\lambda = -\log\beta/\mu$. If $a>0$, $\mu>0$, the expected payoff is of the form $W(x_1,x_2) = -\beta^{x_2} W(x_1)$, where $W(x_1)$ satisfies

$$\tfrac{1}{2}aW'' + \mu W' + (\log\beta)W = 0 \tag{3.5}$$

and $W(0) = -1$. To see what condition to impose at $-\infty$ consider the problem as a limit of problems on the interval $[-M,0]$, $M\to\infty$. So we consider (3.5) with $W(0)=-1$ and $W(-M)=-k_M$, with k_M chosen appropriately. Since in the limiting problem the process will hit zero with probability one and receive no help from the boundary on the left, it is appropriate to choose $k_M=0$; (actually all choices of $k_M\ge 0$ which do not grow too rapidly with M will lead to the same limiting result). This limiting procedure gives us

$$W(x_1) = -e^{\lambda x_1}, \tag{3.6}$$

for $\mu \ge \sqrt{2a \log\beta}$, where $\lambda = \dfrac{-\mu + \sqrt{\mu^2-2a \log\beta}}{a}$ is the maximal root of the equation

$$f(\nu) = \tfrac{1}{2}a\nu^2 + \mu\nu + \log\beta = 0 . \tag{3.7}$$

If

$$\mu < \sqrt{2a \log\beta} , \tag{3.8}$$

the roots of (3.7) are not real and the expected payoff from the constant strategy $\mu(t)\equiv\mu$ and $\sigma(t)\equiv\sigma$ is $-\infty$. In every case, the expected payoff $W(x_1,x_2)$ is given by (3.4), where

$$(3.9)\qquad \lambda = \lambda(\mu,a,\beta) = \begin{cases} \dfrac{-\mu + \sqrt{\mu^2-2a\log\beta}}{a}, & \mu \geq \sqrt{2a\log\beta} \\ \dfrac{-\log\beta}{\mu}, & a=0,\ \mu>0 \\ -\infty, & \text{otherwise.} \end{cases}$$

Condition (3.8) holds if and only if (μ,a) lies on the left of the semi-parabola

$$(3.10)\qquad p:\quad \mu^2 = 2a\log\beta,\quad \mu>0$$

in the (μ,a)-plane. Furthermore all points on the same line segment

$$\ell_\lambda:\ a = \frac{-2}{\lambda}\mu - \frac{2\log\beta}{\lambda^2},\ 0 \leq a \leq \frac{\mu^2}{2\log\beta}$$

give rise to constant strategies with the same expected payoff $-\beta^{x_2}e^{-\lambda x_1}$, $-\infty < \lambda < 0$. Observe that ℓ_λ is the line segment connecting the point $(\frac{-\log\beta}{\lambda},0)$ on the μ-axis to the point $(\frac{-2\log\beta}{\lambda}, \frac{2\log\beta}{\lambda^2})$ on p; at the latter point p and ℓ_λ are tangential. Now set

$$(3.11)\qquad \lambda^* = \lambda^*(\beta) = \sup\{\lambda(\mu,\sigma^2,\beta):\ (\mu,\sigma)\in \mathcal{S}\}.$$

Notice that, for $\lambda = \lambda(\mu,a,\beta) > -\infty$, the function $f(\nu)$ given in

(3.7) is increasing to the right of $\nu=\lambda$. Also, $f(\lambda) = 0$ and $\lambda^* \geq \lambda$. Hence

$$(3.12) \qquad \tfrac{1}{2}\sigma^2\nu^2 + \mu\nu + \log\beta \geq 0, \quad (\mu,\sigma)\in\mathcal{S}, \quad \lambda^* \leq \nu.$$

We define

$$(3.13) \qquad I = \inf_{\varepsilon>0} \sup \{\mu+\varepsilon\sigma^2 : (\mu,\sigma) \in \mathcal{S}\}$$

and observe that the condition

$$(3.14) \qquad I < \infty$$

holds if and only if $\mathcal{S}$ is contained below some line with negative slope.

Lemma 3.1. Assume $I = \infty$. Then $V(x_1,x_2) = -\beta^{x_2}$.

Proof. The assumption allows one to specify, for any $x_1<0$ and $\varepsilon>0$ an $X\in\Sigma(x_1)$ with $ET<\varepsilon$. This was shown in [2]. This means that for $x_1<0$ and $\varepsilon>0$ it is possible to find $X^{(1)}\in\Sigma(x_1)$ with $P[T>\varepsilon] < 1/2$. Next, we find $X^{(2)}\in\Sigma(X^{(1)}(\varepsilon))$ such that $P[T>\frac{\varepsilon}{2}] < 2^{-2}$. Then $X\in\Sigma(x_1)$ is constructed as follows: X agrees with $X^{(1)}$ up to time ε; if $X^{(1)}_\varepsilon\neq 0$, then $X_{\varepsilon+t} = X^{(2)}_t$ for $0 \leq t \leq \varepsilon/2$; etc. Then for the process X, $E\beta^T \leq \beta^{2\varepsilon}$. Since ε is arbitrary the Lemma follows by (3.2). ∎

In view of Lemma 3.1 and the considerations preceding it, one might hope that if $I < \infty$,

$$V(x_1,x_2) = -\beta^{x_2}e^{\lambda^* x_1}.$$

We will now prove that this is indeed the case.

Some difficulties occur where points on the semi-parabola p introduced in (3.10) lie on the boundary of $\mathcal{S}$ without belonging to $\mathcal{S}$. To deal with these we choose a sequence (β_n) of positive reals increasing up to β. Use (3.9) to define

$$\lambda_n(\mu,\sigma^2) = \lambda(\mu,\sigma^2,\beta_n) \tag{3.15}$$

and set

$$\lambda_n^* = \lambda^*(\beta_n) = \sup\{\lambda_n(\mu,\sigma^2) : (\mu,\sigma) \in \mathcal{S}\}. \tag{3.16}$$

Note $\lambda_n^* \le 0$ and λ_n^* decreases as a function of n.
Finally let

$$\lambda^* = \inf_n \lambda_n^* \tag{3.17}$$

which is consistent with (3.11).

<u>Theorem 3.2</u> (a) If $I < \infty$ and $\lambda^* > -\infty$ then

$$V(x_1,x_2) = -\beta^{x_2} e^{\lambda^* x_1}.$$

(b) If $I < \infty$ and $\lambda^* = -\infty$ then

$$V(x_1,x_2) = -\infty.$$

(c) If $I = \infty$ then

$$V(x_1,x_2) = -\beta^{x_2}$$

(that is $E[\beta^T]$ can be made arbitrarily close to 1).

The proof of the Theorem will involve the use of Proposition 1.1. We will construct a sequence of functions $Q_n(x_1,x_2) = \beta^{x_2} Q_n(x_1)$. Let us concentrate on the second factor and write simply x for x_1. Consider the inequality

$$\tfrac{1}{2}\sigma^2 Q_n''(x) + \mu Q_n'(x) + (\log\beta) Q_n(x) \le 0. \tag{3.18}$$

Let

$$U_n(x) = \frac{Q_n'(x)}{\lambda_n^* Q_n(x)} . \tag{3.19}$$

Then $U_n'(x) = \dfrac{1}{\lambda_n^*}\left[\dfrac{Q_n''(x)}{Q_n(x)} - \left[\dfrac{Q_n'(x)}{Q_n(x)}\right]^2\right]$ and so

$U_n'(x) + \lambda_n^*(U_n(x))^2 = \dfrac{1}{\lambda_n^*}\,\dfrac{Q_n''(x)}{Q_n(x)}$. If Q_n is negative, the inequality (3.18) holds if and only if

$$\frac{1}{2}\,\sigma^2\lambda_n^{*2}(U_n(x))^2 + \mu\lambda_n^* U_n(x) + \log\beta + \tfrac{1}{2}\sigma^2\lambda_n^* U_n'(x) \ge 0. \tag{3.20}$$

Our plan is to define an appropriate sequence of functions U_n so that (3.20) holds for all (μ,σ) in $\mathcal{S}$. Then, using the relation (3.19), we transform U_n into Q_n and obtain inequality (3.18) for all $(\mu,\sigma) \in \mathcal{S}$.

The next lemma involves a construction of the functions U_n.

<u>Lemma 3.3</u>. Let $\{k_n : n \ge 1\}$ be positive constants. There exists a sequence $\{U_n : u \ge 1\}$ of real functions with domain $(-\infty,1)$ such that

(i) U_n is continuously differentiable, $n \ge 1$

(ii) $0 < U_n(x) \leq 1, \quad x < 1, \quad n \geq 1$

(iii) $\lim_{n\to\infty} U_n(x) = 1, \quad x \leq 0$

(iv) $\int_{-\infty}^{0} U_n(x)\,dx < \infty$

(v) $0 < U_n'(x) \leq k_n U_n(x), \quad n \geq 1, \quad x \leq 0.$

<u>Proof</u>: For each n define

$$U_n(x) = \begin{cases} 1, & -n < x < 1 \\ c_n \int_{-\infty}^{x+n} y\, e^{k_n y}\, dy, & x < -n. \end{cases}$$

where $c_n = \left[\int_{-\infty}^{0} y e^{k_n y} dy\right]^{-1}$.

Then (i) - (iii) are easily checked. Integration by parts gives

$$U_n(x) = \frac{c_n}{k_n} e^{k_n(x+n)} (x+n-\frac{1}{k_n}), \quad x < -n.$$

The last identity implies (iv) and also

$$U_n(x) \geq \frac{c_n}{k_n} e^{k_n(x+n)} (x+n), \quad x < -n.$$

That is,

$$U_n(x) \geq \frac{1}{k_n} U_n'(x) \; ,$$

establishing property (v). ∎

Assume now that I < ∞ (see 3.13), and $\lambda^* > -\infty$. Then there exist positive constants ρ and M such that

(3.21) for all (μ,σ) in $\mathcal{S}$, $\mu + \rho\sigma^2 \leq M$.

For each positive integer n let

$$(3.22) \qquad k_n = \begin{cases} \min\left\{2\rho, \dfrac{-2\rho \log\frac{\beta}{\beta_n}}{M\lambda_n^*}\right\}, & \lambda_n^* < 0 \\ 2\rho, \quad \lambda_n^* = 0. \end{cases}$$

With these constants k_n, let U_n be functions satisfying the conditions of Lemma 3.3.

<u>Lemma 3.4.</u> Assume $\lambda^* > -\infty$, (3.21), and let $\{U_n : n \geq 1\}$ be the sequence of functions specified above. Then for $n \geq 1$, $x<0$ and $(\mu,\sigma) \in \mathcal{S}$, inequality (3.20) holds.

<u>Proof</u>. The assertion is clear for $\lambda^*=0$. So we suppose $\lambda^*<0$. By properties (ii) and (v) of Lemma 3.3 and the definition of k_n in (3.22) we have

$$(3.23) \qquad U_n'(x) \leq \frac{-2\rho \log(\beta/\beta_n)}{M\lambda_n^*} .$$

Let $g_n(x)$ denote the expression on the left of (3.20).

<u>Case 1</u>: $\mu \geq 0$, $\lambda_n^* < 0$. Use (3.21) to obtain

$$(3.24) \qquad \rho\sigma^2 \leq \mu + \rho\sigma^2 \leq M.$$

Now write

$$(3.25) \qquad g_n(x) = [\tfrac{1}{2}\sigma^2\lambda_n^{*2}(U_n(x))^2 + \mu\lambda_n^* U_n(x) + \log \beta_n] + \log \frac{\beta}{\beta_n} + \tfrac{1}{2}\sigma^2\lambda_n^* U_n'(x) .$$

Since $\lambda_n^* \leq \lambda_n^* U_n(x) \leq 0$, the expression in brackets is greater than or equal to zero by (3.12). Then (3.24) and (3.23) show that the sum of the last two terms in (3.25) is non-negative. Hence $g_n(x) \geq 0$, as desired.

Case 2: $\mu < 0$, $\lambda_n^* < 0$. Use (3.21) to obtain

$$(3.26)\quad g_n(x) \geq \frac{1}{2}\sigma^2(\lambda_n^* U_n(x))^2 + \mu\lambda_n^* U_n(x) + \log\beta + \frac{\lambda_n^*(M-\mu)}{2\rho}U_n'(x)$$

$$\geq \mu\lambda_n^*[U_n(x) - \frac{1}{2\rho}U_n'(x)] + \frac{M\lambda_n^*}{2\rho}U_n'(x) + \log\beta.$$

By properties (ii) and (v) of Lemma 3.3 and the definition of k_n in (3.22),

$$U_n(x) - \frac{1}{2\rho}U_n'(x) \geq 0$$

and

$$U_n'(x) \leq \frac{-2\rho\log\beta}{M\lambda_n^*}.$$

Using these inequalities in (3.26) and recalling $\mu<0$, $\lambda_n^*<0$ one concludes $g_n(x) \geq 0$. ∎

The next lemma will be used to create processes whose payoffs approach $-\beta^{x_2}e^{\lambda x_1}$.

Lemma 3.5. If $I < \infty$, $\lambda^* > -\infty$, then there exists a sequence $\{(\mu_k,\sigma_k): k\geq 1\}$ of elements of $\mathcal{S}$ such that the sequence converges in $\mathbb{R}^2$ to a limit $(\bar{\mu},\bar{\sigma})$ and

$$(3.27)\qquad \frac{1}{2}(\lambda^*\bar{\sigma})^2 + \lambda^*\bar{\mu} + \log\beta = 0.$$

(Note $(\bar{\mu},\bar{\sigma})$ might not lie in $\mathcal{S}$).

Proof. Using the definition of λ^*, there is a strictly increasing sequence $\{n_k: k\geq 1\}$ of positive integers and a sequence $\{(\mu_k,\sigma_k): k\geq 1\}$ of elements of $\mathcal{S}$ such that

$$\lim_{k\to\infty} \lambda_{n_k}(\mu_k,\sigma_k^2) = \lambda^* .$$

By passing to a subsequence if necessary, we can assume that either (case 1) $\sigma_k>0$ and $\mu_k \geq \sigma_k\sqrt{2 \log \beta_{n_k}}$ for each k or (case 2) $\sigma_k=0$ and $\mu_k>0$ for each k. Since $I<\infty$, the set $\{(\mu,\sigma):(\mu,\sigma)\in\mathcal{S}, \mu\geq 0\}$ is a bounded subset of $\mathbb{R}^2$. Thus by passing to a further subsequence, we can assume the sequence converges in $\mathbb{R}^2$.

<u>Case 1</u>. $\sigma_k>0$ and $\mu_k \geq \sigma_k\sqrt{2 \log\beta_{n_k}}$ for each k. By definition,

$$\lambda_{n_k}(\mu_k,\sigma_k) = \frac{-\mu_k + \sqrt{\mu_k^2-2\sigma_k^2\log\beta_{n_k}}}{\sigma_k^2}, \text{ and so for each k,}$$

$$\frac{1}{2}\,[\lambda_{n_k}(\mu_k,\sigma_k^2)]^2\sigma_k{}^2 + [\lambda_{n_k}(\mu_k,\sigma_k^2)]\mu_k + \log \beta_{n_k} = 0.$$

Taking the limit as $k\to\infty$, we get (3.27).

<u>Case 2</u>. $\sigma_k = 0$ and $\mu_k > 0$ for each k. By definition,

$$\lambda_{n_k}(\mu_k,\sigma_k^2) = \frac{-\log \beta_{n_k}}{\mu_k} .$$

Take limits to get $\lambda^*\bar{\mu} = -\log \beta$ and $\bar{\sigma} = 0$, and (3.27) follows. ∎

<u>Proof of the theorem</u>. (a) Define $Q\colon (-\infty,0]\times\mathbb{R}\longrightarrow\mathbb{R}$ by

$$Q(x_1,x_2) = -\beta^{x_2} e^{\lambda^* x_1} .$$

To show $Q \leq V$, begin by fixing $x_1 \leq 0$ and $x_2 \in \mathbb{R}$. By Lemma 3.5, there is a sequence $\{(\mu_k,\sigma_k)\colon k \geq 1\}$ in $\mathcal{S}$ satisfying relation (3.27). Let $\varepsilon > 0$ and use (3.27) to create two functions $\hat{\mu}\colon[0,\infty)\longrightarrow\mathbb{R}$ and $\hat{\sigma}\colon[0,\infty)\longrightarrow\mathbb{R}$ such that

$$(\hat{\mu}(s),\hat{\sigma}(s)) \in \{(\mu_k,\sigma_k): k \geq 1\}$$

for each $s \geq 0$,

$$\int_0^\infty \left|\frac{1}{2}\lambda^{*2}[\hat{\sigma}(s)]^2 + \lambda^*\hat{\mu}(s) + \log\beta\right| ds < \log(1+\varepsilon), \tag{3.28}$$

and

$$\inf\{\hat{\mu}(s): s\geq 0\} > 0 . \tag{3.29}$$

Let X be the process given by

$$X_1(t) = x_1 + \int_0^t \hat{\mu}(s)ds + \int_0^t \hat{\sigma}(s)dW_s$$

$$X_2(t) = x_2 + t$$

for $t\leq T$, where $T = \inf\{t: X_1(t) = 0\}$, and $X(t) = X(T)$ for $t\geq T$. Notice that (3.29) guarantees that the stopping time T is finite almost-surely.

The aim is to show that X has payoff near $-\beta^{x_2}e^{\lambda^* x_1}$. Define the process Y by

$$Y_t = \exp\left[\lambda^*\left(X_1(t) - \int_0^t \hat{\mu}(s)ds\right) - \frac{\lambda^{*2}}{2}\int_0^t \hat{\sigma}^2(s)ds\right] .$$

That is,

$$Y_t = \exp\left[\lambda^* x_1 + \lambda^*\int_0^t \hat{\sigma}(s)dW_s - \frac{\lambda^{*2}}{2}\int_0^t \hat{\sigma}^2(s)ds\right] .$$

By Ito's Formula,

$$Y_t = \exp\left[\lambda^* x_1\right] + \lambda^*\int_0^t Y_s\hat{\sigma}(s)dW_s ,$$

and so $\{Y_t\}$ is a local martingale. Further, since $\{Y_t\}$ is non-negative, it follows (see Dellacherie and Meyer [1], VI.29) that $EY_T \leq EY_0$. That is,

$$(3.30)\qquad E\left\{\exp\left[-\int_0^T \{\lambda^*\hat{\mu}(s) + \tfrac{1}{2}\lambda^{*2}\hat{\sigma}^2(s)\}ds\right]\right\} \le \exp\left[\lambda^* x_1\right] .$$

Using (3.28), (3.30), and the fact that $X_1(T) = 0$ a.s.,

$$E\beta^T = E[\exp[(\log \beta)T]] \le (1+\varepsilon)\exp[\lambda^* x_1] .$$

Thus X is available at (x_1,x_2) because $E\beta^T<\infty$, and X has payoff at least $-\beta^{x_2}(1+\varepsilon)\exp[\lambda^* x_1] = (1+\varepsilon)Q(x_1,x_2)$. We conclude $Q \le V$.

To show $Q \ge V$ we would like to apply Proposition 1.1. However, Q does not satisfy (iii) of that proposition. So we construct a sequence $\{Q_n\}$ converging pointwise to Q such that Proposition 1.1 applies to each Q_n.

For each $n \ge 1$, define $Q_n:(-\infty,1)\times\mathbb{R}\to\mathbb{R}$ by

$$(3.31)\qquad Q_n(x_1,x_2) = -\beta^{x_2}\exp\left[-\int_{x_1}^0 \lambda_n^* U_n(y)\,dy\right] .$$

Notice that $\lim_{n\to\infty} Q_n(x) = Q(x)$ for each x in $F = (-\infty,0]\times\mathbb{R}$ because of properties (ii) and (iii) of Lemma 3.3 and the dominated convergence theorem. Also, each Q_n is twice continuously differentiable because of property (i) of Lemma 3.3.

Now verify the conditions of Proposition 1.1 with Q_n $(n\ge 1)$ in place of Q. Condition (i) is immediate. For condition (ii), let $x = (x_1,x_2) \in F^\circ$ and check that for each (μ,σ) in $\mathcal{S}$,

$$(3.32)\qquad \left[\frac{\partial}{\partial x_2} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_1}\right] Q_n(x_1,x_2)$$

$$= Q_n(x_1,x_2)\left(\tfrac{1}{2}\sigma^2\lambda_n^{*2}[U_n(x_1)]^2 + \tfrac{1}{2}\sigma^2\lambda_n^* U_n'(x_1) + \mu\lambda_n^* U(x_1) + \log \beta\right).$$

Use Lemma 3.4 and the fact that $Q_n(x_1,x_2) \le 0$ to show that the expression (3.32) is non-positive. For condition (iii), let $x \in \Sigma_C(x)$ and use (3.31) and (ii) and (iv) of Lemma 3.3 to see that

$$(3.33) \qquad Q_n(X_1(t),X_2(t)) \ \ge\ -C\beta^{X_2(t)} \ \ge\ -C\beta^{x_2+T}$$

where C is a constant satisfying $0 < C < \infty$. Now $E(\beta^T) < \infty$ for each $X \in \Sigma_C(x)$, and hence condition (iii) follows from (3.33). Thus Proposition 1.1 shows that $Q_n \ge V$ for each n and hence that $Q \ge V$. This completes the proof of (a) of the theorem.

(b). We reduce the result to (a). The hypothesis for (b) is that $I < \infty$ and $\lambda^* = -\infty$. Let $\varepsilon > 0$ and consider a new problem based on the set

$$\mathcal{S}_\varepsilon = \mathcal{S} \cup \{(\varepsilon,0)\} .$$

The quantity corresponding to λ^* for the new problem is

$$\lambda^*_\varepsilon = \frac{-\log \beta}{\varepsilon} .$$

Thus part (a) can be applied to obtain the value function

$$V_\varepsilon(x_1,x_2) = -\beta^{x_2 - \frac{x_1}{\varepsilon}} \quad \text{for } x_1 \le 0,\ x_2 \in \mathbb{R}.$$

Clearly $V(x_1,x_2) \ge V_\varepsilon(x_1,x_2) \to -\infty$ as $\varepsilon \to 0$, and so the proof of (b) is finished.

(c). This was proved in Lemma 3.1. ∎

<u>Example</u>. Inflated, continuous-time red-and-black. The problem considered here is the same as that in the example of section 2 except that $\beta > 1$ and the player seeks to minimize $E\beta^T$. (Imagine

a borrower of \$1.00 who must pay back $\$\beta^T$ if the loan is repaid at time T.) After the change of coordinates (2.15), the control set $\mathcal{S}$ is given by (2.16) and is obviously bounded so that $I < \infty$. The quantity $\lambda(\mu,\sigma^2) = \lambda(s)$ is given by (2.17) if $s > 0$ and $\mu > \sqrt{2\sigma^2 \log\beta}$. Substitute $\mu = s\mu_0 - \frac{1}{2}s^2\sigma_0^2$, $\sigma = s\sigma_0$ and the latter condition reduces to $s \le M$, $M = 2\mu_0/\sigma_0^2 - 2\sqrt{2(\log\beta)}/\sigma_0$. To reiterate, $\lambda(s)$ is given by (2.17) if $0 < s \le M$ and $\lambda(s) = -\infty$ if not. Notice that, in the subfair case ($\mu_0 \le 0$), $M < 0$, and consequently $\lambda^* = -\infty$ and, by Theorem 3.2, $V \equiv -\infty$. In the superfair case ($\mu_0 > 0$), one shows that $\lambda'(s) > 0$ if and only if $0 < s < M \wedge c$ where $c = \mu_0/\sigma_0^2 - 2(\log\beta)/\mu_0$. Thus $\lambda^* = \lambda(s^*)$ where $s^* = (M \wedge c \wedge 1) \vee 0$, and V is given by Theorem 3.2.

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